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Quantum correlations and number theory

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Abstract

We study the spin-1/2 Heisenberg *XXX* antiferromagnet for which the spectrum of the Hamiltonian was found by Bethe in 1931. We study the probability of the formation of ferromagnetic string in the antiferromagnetic ground state, which we call emptiness formation probability $P(n)$. This is the most fundamental correlation function. We prove that, for short strings, it can be expressed in terms of the Riemann zeta function with odd arguments, logarithm $\ln 2$ and rational coefficients. This adds yet another link between statistical mechanics and number theory. We have obtained an analytical formula for $P(5)$ for the first time. We have also calculated $P(n)$ numerically by the density matrix renormalization group. The results agree quite well with the analytical results. Furthermore, we study the asymptotic behaviour of $P(n)$ at finite temperature by quantum Monte Carlo simulation. This also agrees with our previous analytical results.

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1. Introduction

Recently, considerable progress has been achieved in the exact calculation of the correlation functions in the spin-1/2 Heisenberg *XXZ* chain [1–11]. Bethe [12] discovered his ansatz, while diagonalizing the *XXX* Hamiltonian. The most important features of exactly solvable models, such as two-body reducibility of dynamics, were first discovered for this model. We believe that the antiferromagnetic Heisenberg *XXX* chain is one of the most fundamental exactly solvable models. Recently we have developed a new method of evaluating the multi-integral representation of correlation functions [9, 10]. We study the most fundamental correlation function of the model, which we call the emptiness formation probability (EFP) $P(n)$. This was first introduced in [3]

$$P(n) = \langle \mathrm{GS}| \prod_{j=1}^{n} P_j |\mathrm{GS}\rangle, \tag{1.1}$$

where $P_j = S_j^z + \frac{1}{2}$ is the projector on the state with the spin up in the jth lattice site. $|\mathrm{GS}\rangle$ is the antiferromagnetic ground state in the thermodynamic limit constructed by Hulthén [13]. $P(n)$ is the probability of the formation of a ferromagnetic string of length n in $|\mathrm{GS}\rangle$.

The Hamiltonian of the Heisenberg *XXX* chain is given by

$$H = J\sum_{j=1}^{N}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z - \tfrac{1}{4}), \tag{1.2}$$

where the coupling constant J is positive for the antiferromagnet.

In [9, 10], we wrote the Hamiltonian (1.2) in terms of the Pauli matrices. It corresponds to $J = 4$. Note that the Hamiltonian (1.2) annihilates the ferromagnetic states $|\uparrow\cdots\uparrow\rangle$ (or $|\downarrow\cdots\downarrow\rangle$).

The first cases $P(3)$ and $P(4)$ were calculated by means of the multi-integral representation in [9, 10]. In this paper, we present a new analytic formula for $P(5)$

$$\begin{aligned} P(5) = {} & \tfrac{1}{6} - \tfrac{10}{3}\ln 2 + \tfrac{281}{24}\zeta(3) - \tfrac{45}{2}\ln 2\zeta(3) - \tfrac{489}{16}\zeta(3)^2 \\ & - \tfrac{6775}{192}\zeta(5) + \tfrac{1225}{6}\ln 2\zeta(5) - \tfrac{425}{64}\zeta(3)\zeta(5) - \tfrac{12\,125}{256}\zeta(5)^2 \\ & + \tfrac{6223}{256}\zeta(7) - \tfrac{11\,515}{64}\ln 2\zeta(7) + \tfrac{42\,777}{512}\zeta(3)\zeta(7) = 2.011\,725\,953\times 10^{-6}, \end{aligned} \tag{1.3}$$

where $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \qquad \text{for } \Re(s) > 1. \tag{1.4}$$

$P(5)$ is expressed in terms of $\ln 2$, $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$ with rational coefficients. In fact, it was conjectured [10]:
'$P(n)$ is always expressed in terms of logarithm $\ln 2$, Riemann zeta functions $\zeta(2k+1)$ with odd argument and rational coefficients'.

This means that all values of $P(n)$ are different transcendental numbers [14].

For comparison, let us list previous results for $P(n)$:

$$P(1) = \tfrac{1}{2} = 0.5, \tag{1.5}$$

$$P(2) = \tfrac{1}{3} - \tfrac{1}{3}\ln 2 = 0.102\,284\,273, \tag{1.6}$$

$$P(3) = \tfrac{1}{4} - \ln 2 + \tfrac{3}{8}\zeta(3) = 0.007\,624\,158, \tag{1.7}$$

$$\begin{aligned} P(4) &= \tfrac{1}{5} - 2\ln 2 + \tfrac{173}{60}\zeta(3) - \tfrac{11}{6}\ln 2\zeta(3) - \tfrac{51}{80}\zeta^2(3) - \tfrac{55}{24}\zeta(5) + \tfrac{85}{24}\ln 2\zeta(5) \\ &= 0.000\,206\,270. \end{aligned} \tag{1.8}$$

Let us mention that in contrast with $P(1)$, $P(2)$, $P(3)$ (which do not contain non-linear terms of the Riemann zeta function) the values $P(4)$ and $P(5)$ do contain non-linear terms in the Riemann zeta function. It might be instructive to express formulae (1.3) and (1.8) for $P(5)$ and $P(4)$ in a linear form by introducing the multiple zeta values [15]

$$\zeta(k_1, k_2, \ldots, k_m) = \sum_{n_1 > n_2 > \cdots > n_m > 0} n_1^{-k_1} n_2^{-k_2} \ldots n_m^{-k_m}. \tag{1.9}$$

The length (or depth) of this multiple zeta value is equal to m and the level (or weight) is equal to $k_1 + k_2 + \cdots + k_m$. The results are

$$\begin{aligned} P(4) = {} & \tfrac{1}{5} - 2\ln 2 + \tfrac{173}{60}\zeta(3) - \tfrac{11}{6}\ln 2\zeta(3) - \tfrac{55}{24}\zeta(5) + \tfrac{85}{24}\ln 2\zeta(5) \\ & - \tfrac{51}{10}\zeta(3,3) + \tfrac{153}{80}\zeta(4,2), \end{aligned} \tag{1.10}$$

$$\begin{aligned} P(5) = {} & \tfrac{1}{6} - \tfrac{10}{3}\ln 2 + \tfrac{281}{24}\zeta(3) - \tfrac{45}{2}\ln 2\zeta(3) - \tfrac{6775}{192}\zeta(5) \\ & + \tfrac{1225}{6}\ln 2\zeta(5) + \tfrac{6223}{256}\zeta(7) - \tfrac{11\,515}{64}\ln 2\zeta(7) \end{aligned}$$

$$- \tfrac{489}{2}\zeta(3,3) + \tfrac{1467}{16}\zeta(4,2) - \tfrac{12\,495}{128}\zeta(4,4) - \tfrac{85}{64}\zeta(5,3) - \tfrac{425}{128}\zeta(6,2) + \tfrac{487\,037}{512}\zeta(5,5) + \tfrac{584\,425}{1024}\zeta(6,4) + \tfrac{596\,647}{1024}\zeta(7,3) + \tfrac{42\,777}{1024}\zeta(8,2). \qquad (1.11)$$

Here we used the following identities:

$$\begin{aligned} \zeta(3)^2 &= 8\zeta(3,3) - 3\zeta(4,2), \\ \zeta(3)\zeta(5) &= \tfrac{147}{10}\zeta(4,4) + \tfrac{1}{5}\zeta(5,3) + \tfrac{1}{2}\zeta(6,2), \\ \zeta(5)^2 &= 22\zeta(5,5) + 10\zeta(6,4) + 10\zeta(7,3), \\ \zeta(3)\zeta(7) &= \tfrac{167}{7}\zeta(5,5) + \tfrac{25}{2}\zeta(6,4) + \tfrac{177}{14}\zeta(7,3) + \tfrac{1}{2}\zeta(8,2). \end{aligned} \qquad (1.12)$$

These identities can be derived using the recurrent relations (4.1) and (4.5)–(4.7) of [16]. Let us note that the Drinfeld associator is also related to multiple zeta values in a linear way [17].

Since ln 2 in the above formulae looks somehow isolated, we believe that it seems to be more appropriate to express $P(n)$ in terms of the alternating zeta series (the value of polylogarithm at the root of unity)

$$\zeta_a(s) = \sum_{n>0} \frac{(-1)^{n-1}}{n^s} = -\mathrm{Li}_s(-1). \qquad (1.13)$$

Here $\mathrm{Li}_s(x)$ is the polylogarithm. The alternating zeta series is related to the Riemann zeta function as follows

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\zeta_a(s). \qquad (1.14)$$

This formula is valid for $s \neq 1$. In contrast with the zeta function (which has the pole when $s \to 1$), the alternating zeta has a limit as $s \to 1$

$$\zeta_a(1) = \ln 2. \qquad (1.15)$$

Using formulae (1.3)–(1.8), (1.14) and (1.15), one can obtain the five first values of $P(n)$ expressed via the alternating zeta series

$$\begin{aligned} P(1) &= \tfrac{1}{2}, \\ P(2) &= \tfrac{1}{3}\{1 - \zeta_a(1)\}, \\ P(3) &= \tfrac{1}{4}\{1 - 4\zeta_a(1) + 2\zeta_a(3)\}, \\ P(4) &= \tfrac{1}{5}\big\{1 - 10\zeta_a(1) + \tfrac{173}{9}\zeta_a(3) - \tfrac{110}{9}\zeta_a(5) - \tfrac{110}{9}\zeta_a(1)\zeta_a(3) \\ &\quad + \tfrac{170}{9}\zeta_a(1)\zeta_a(5) - \tfrac{17}{3}\zeta_a^2(3)\big\}, \\ P(5) &= \tfrac{1}{6}\big\{1 - 20\zeta_a(1) + \tfrac{281}{3}\zeta_a(3) - \tfrac{1355}{6}\zeta_a(5) + \tfrac{889}{6}\zeta_a(7) \\ &\quad - 180\zeta_a(1)\zeta_a(3) + \tfrac{3920}{3}\zeta_a(1)\zeta_a(5) - \tfrac{3290}{3}\zeta_a(1)\zeta_a(7) - \tfrac{170}{3}\zeta_a(3)\zeta_a(5) \\ &\quad + 679\zeta_a(3)\zeta_a(7) - 326\zeta_a^2(3) - \tfrac{970}{3}\zeta_a^2(5)\big\}. \end{aligned} \qquad (1.16)$$

The values of the Riemann zeta function at odd arguments appear in several places in theoretical physics, not to mention in pure mathematics. The transcendental number $\zeta(3)$ first appeared in the expression for correlation functions in papers by Takahashi [19, 20]. He evaluated the second neighbour correlation

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+2} \rangle = \tfrac{1}{4} - 4\ln 2 + \tfrac{9}{4}\zeta(3). \qquad (1.17)$$

This was obtained from the $1/U$ expansion of the ground-state energy for the half-filled Hubbard chain (see also another derivation in [21]). We remark that the expression (1.7) for $P(3)$ can be extracted from (1.17).

Now let us discuss the asymptotic behaviour of $P(n)$ when n is large. At zero temperature we believe that $P(n)$ should show a Gaussian decay as n tends to infinity [9, 10]. In order to prove this mathematically we have to obtain a general formula for $P(n)$ which has not been achieved yet. On the other hand, we can calculate $P(n)$ by numerical means and confirm our analytical results. We have applied the density matrix renormalization group (DMRG) method [22–24] and obtained more numerical values of $P(n)$. The result is given at the end of section 2. At finite temperature it was shown that $P(n)$ decays exponentially [9]. This time we can employ the quantum Monte Carlo (QMC) simulation [25] to calculate $P(n)$ numerically. In section 3 we confirm that $P(n)$ exhibits an exponential decay at finite temperature and confirm again our analytical expression. Let us note that this numerical approach was successfully applied to the *XX* model in [11].

2. $P(n)$ at zero temperature

The integral representation of $P(n)$ for the *XXX* chain was obtained in [3] based on the vertex operator approach [2]

$$P(n) = \int_C \frac{\mathrm{d}\lambda_1}{2\pi \mathrm{i}\lambda_1} \int_C \frac{\mathrm{d}\lambda_2}{2\pi \mathrm{i}\lambda_2} \cdots \int_C \frac{\mathrm{d}\lambda_n}{2\pi \mathrm{i}\lambda_n} \prod_{a=1}^{n} \left(1 + \frac{\mathrm{i}}{\lambda_a}\right)^{n-a} \left(\frac{\pi\lambda_a}{\sinh \pi\lambda_a}\right)^n \times \prod_{1\leqslant j<k\leqslant n} \frac{\sinh \pi(\lambda_k - \lambda_j)}{\pi(\lambda_k - \lambda_j - \mathrm{i})}. \tag{2.1}$$

The contour C in each integral is parallel to the real axis with the imaginary part between 0 and $-\mathrm{i}$. Below, we sketch the evaluation of the integral for $P(5)$. It can be written in the following form

$$P(5) = \prod_{j=1}^{5} \int_C \frac{\mathrm{d}\lambda_j}{2\pi \mathrm{i}} U_5(\lambda_1, \ldots, \lambda_5) T_5(\lambda_1, \ldots, \lambda_5), \tag{2.2}$$

where

$$U_5(\lambda_1, \ldots, \lambda_5) = \pi^{15} \frac{\prod_{1\leqslant k<j\leqslant 5} \sinh \pi(\lambda_j - \lambda_k)}{\prod_{j=1}^{5} \sinh^5 \pi\lambda_j}, \tag{2.3}$$

and

$$T_5(\lambda_1, \ldots, \lambda_5) = \frac{\prod_{j=1}^{5} \lambda_j^{j-1} (\lambda_j + \mathrm{i})^{5-j}}{\prod_{1\leqslant k<j\leqslant 5} (\lambda_j - \lambda_k - \mathrm{i})}. \tag{2.4}$$

Taking into account the properties of the functions $U_5(\lambda_1, \ldots, \lambda_5)$, one can reduce the integrand $T_5(\lambda_1, \ldots, \lambda_5)$ to the 'canonical form', $T_5^c(\lambda_1, \ldots, \lambda_5)$ as was done for $P(2)$, $P(3)$ and $P(4)$ in [10]

$$T_5^c(\lambda_1, \ldots, \lambda_5) = P_0^{(5)} + \frac{P_1^{(5)}}{\lambda_2 - \lambda_1} + \frac{P_2^{(5)}}{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}, \tag{2.5}$$

where $P^{(0)}$, $P^{(1)}$ and $P^{(2)}$ are polynomials of the integration variables $\lambda_1, \ldots, \lambda_5$. The manifest form of these polynomials is shown in the appendix.

Now, using the methods developed in [10], we can calculate three integrals that contribute into $P(5)$

$$J_0^{(5)} = \prod_{j=1}^{5} \int_C \frac{\mathrm{d}\lambda_j}{2\pi \mathrm{i}} U_5(\lambda_1, \ldots, \lambda_5) P_0^{(5)}(\lambda_1, \ldots, \lambda_5), \tag{2.6}$$

$$J_1^{(5)} = \prod_{j=1}^{5} \int_C \frac{\mathrm{d}\lambda_j}{2\pi \mathrm{i}} U_5(\lambda_1, \ldots, \lambda_5) \frac{P_1^{(5)}(\lambda_1, \ldots, \lambda_5)}{\lambda_2 - \lambda_1}, \tag{2.7}$$

Table 1. DMRG data for $P(n)$ with uncertainties in the final digits.

n	$P(n)$ by DMRG method
2	$1.022\underline{2} \times 10^{-1}$
3	$7.623\underline{8} \times 10^{-3}$
4	$2.06\underline{0} \times 10^{-4}$
5	$2.01\underline{0} \times 10^{-6}$
6	$7.0\underline{5} \times 10^{-9}$
7	$8.8\underline{5} \times 10^{-12}$
8	$3.\underline{7} \times 10^{-15}$

$$J_2^{(5)} = \prod_{j=1}^{5} \int_C \frac{d\lambda_j}{2\pi i} U_5(\lambda_1, \ldots, \lambda_5) \frac{P_2^{(5)}(\lambda_1, \ldots, \lambda_5)}{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}. \tag{2.8}$$

The results are

$$J_0^{(5)} = \tfrac{689}{576}, \tag{2.9}$$

$$J_1^{(5)} = -\tfrac{593}{576} - \tfrac{10}{3}\ln 2 - \tfrac{2773}{384}\zeta(3) - \tfrac{175}{48}\zeta(5) + \tfrac{13\,727}{1024}\zeta(7), \tag{2.10}$$

$$\begin{aligned} J_2^{(5)} = {} & \tfrac{2423}{128}\zeta(3) - \tfrac{45}{2}\ln 2\zeta(3) - \tfrac{489}{16}\zeta(3)^2 - \tfrac{2025}{64}\zeta(5) + \tfrac{1225}{6}\ln 2\zeta(5) - \tfrac{425}{64}\zeta(3)\zeta(5) \\ & - \tfrac{12\,125}{256}\zeta(5)^2 + \tfrac{11\,165}{1024}\zeta(7) - \tfrac{11\,515}{64}\ln 2\zeta(7) + \tfrac{42\,777}{512}\zeta(3)\zeta(7). \end{aligned} \tag{2.11}$$

Summing up these three values we come to our final answer (1.3).

Using the same method we can, in principle, obtain the analytic formula for any $P(n)$. Unfortunately, so far we have not succeeded in calculating $P(n)$ for $n \geqslant 6$. On the other hand, one can estimate the numerical values of $P(n)$ using the DMRG method [22,23]. This method is suitable for studying ground-state properties. We followed the standard algorithm, which can be found in the literature (see [24]). Below, we outline some technical points that are relevant for the simulation precision. We implemented the infinite-system method. We have repeated renormalization 200 times. At each renormalization, we kept, at most, 200 relevant states for a (new) block; namely, we set $m = 200$. The density-matrix eigenvalue $\{w_\alpha\}$ of the remaining bases indicates the statistical weight. We found $w_\alpha > 10^{-10}$; that is, we have retained almost all relevant states with appreciable statistical weight $w_\alpha > 10^{-10}$ through numerical renormalizations. In other words, we have discarded (disregarded) those states with exceedingly small statistical weight $w_\alpha < 10^{-10}$, which may indicate an error in the present simulation.

The obtained data are shown in table 1.

Compared with the exact values (1.3) and (1.5)–(1.8), we see that the DMRG data achieve an accuracy of about three digits up to $P(5)$. Probably the other data, i.e., $P(6), \ldots, P(8)$ also maintain an accuracy of, at least, one or two digits. Then, combining the exact values up to $P(5)$ and the numerical data in table 1, we have made a semi-log plot in figure 1. On the vertical axis we have plotted $\ln P(n)$, and on the horizontal axis we have plotted n^2. From figure 1 we clearly see that the data fall into a straight line suggesting that the asymptotic form of $P(n)$ is governed by the Gaussian form

$$P(n) \sim a^{-n^2}. \tag{2.12}$$

We can read off the Gaussian decay rate a from the slope. Our estimate is $a = 1.6719 \pm 0.0005$. It is an intriguing open problem to associate this number with a certain analytical expression. Dr A G Abanov has confirmed the Gaussian form of the asymptotic expression in the framework of the bosonization technique.

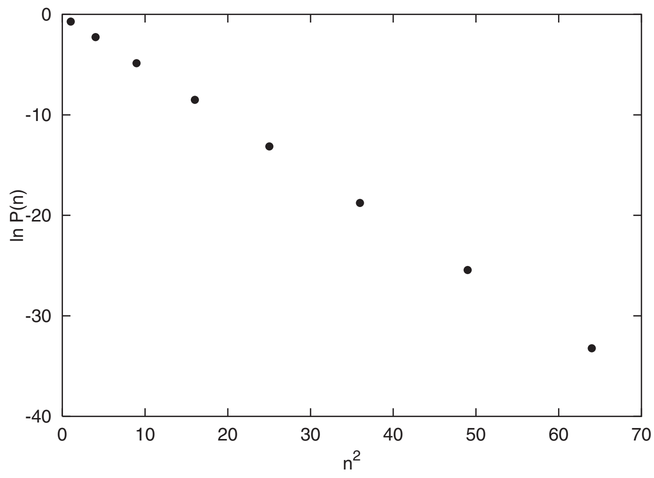

Figure 1. $P(n)$ at zero temperature.

3. $P(n)$ at finite temperature by QMC simulation

At finite temperature T the EFP $P(n)$ is defined by the thermal average

$$P(n) = \frac{\mathrm{Tr}\{\mathrm{e}^{-H/T}\prod_{j=1}^{n} P_j\}}{Z} \qquad Z = \mathrm{Tr}\{\mathrm{e}^{-H/T}\} = \mathrm{e}^{-Nf/T} \tag{3.1}$$

where f is the free energy per site of the system.

It was shown in [9] that $P(n)$ decays exponentially at finite temperature as n goes to infinity

$$P(n) \sim c_0(T)\mathrm{e}^{nf/T}. \tag{3.2}$$

Here $c_0(T)$ is a constant prefactor. Note that in our derivation of (3.2) we have used the fact that our Hamiltonian (1.2) annihilates the ferromagnetic state (all spins up). We confirm this formula by calculating $P(n)$ numerically by QMC simulation [25]. We adopted the continuous-time algorithm [26] with the cluster-flip update [27–29] which is completely free from the Trotter-decomposition error. We treated system sizes up to $N = 128$, and imposed the periodic boundary condition. We performed five million Monte Carlo steps initiated by 0.5 million steps for reaching thermal equilibrium. $P(n)$ is measured over the five million Monte Carlo steps. The semi-log plots of the data for several different temperatures are shown in figure 2. The straight lines (slopes) are the analytic asymptotic formula (3.2). We assumed $c_0(T) = 1$ for simplicity. Actually $c_0(T)$ deviates from unity for low temperature. Anyway we observe that, as $n \to \infty$, the EFP $P(n)$ decays exponentially according to the asymptotic form (3.2). In particular, at sufficiently high temperatures, even for small n, the EFP $P(n)$ is well fitted by (3.2). In contrast, at low temperature, when n becomes small, $P(n)$ may reflect the Gaussian decay at zero temperature and deviates from (3.2). This agrees with the physical picture. Since zero temperature is a critical point, we expect a qualitative asymptotic change in correlation functions.

Finally let us comment on how we can evaluate the free energy per site f from the point of view of the Bethe ansatz. There are now three different integral equations which determine the free energy f:

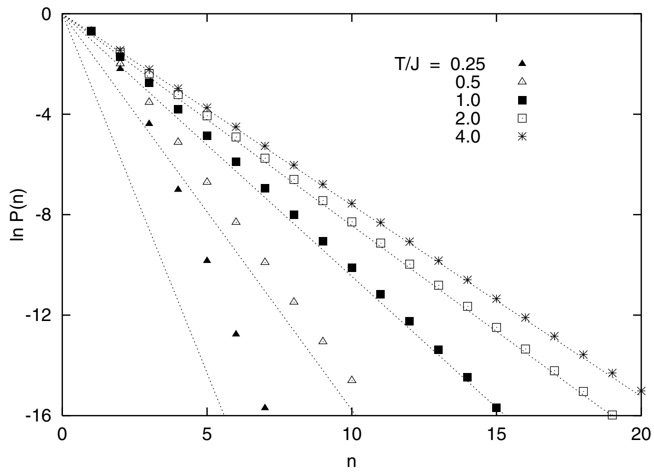

Figure 2. $P(n)$ at finite temperature.

1. the thermodynamic Bethe ansatz (TBA) equations formulated by Takahashi [30] based on the string hypothesis;
2. the non-linear integral equations (NLIE) found by Klümper [31, 32] and Destri and de Vega [33] in the development of the quantum transfer matrix method [34–37];
3. the new integral equation derived by Takahashi [38].

The third integral equation was recently discovered in an attempt to simplify the TBA equations [38]. It also has a close connection with the quantum transfer matrix [39]. The equation is explicitly given by

$$u(x) = 2 + \oint_C \left(\frac{1}{x - y - 2\mathrm{i}} \exp\left[\frac{2J/T}{(y + \mathrm{i})^2 + 1} \right] + \frac{1}{x - y + 2\mathrm{i}} \exp\left[\frac{2J/T}{(y - \mathrm{i})^2 + 1} \right] \right) \frac{1}{u(y)} \frac{\mathrm{d}y}{2\pi \mathrm{i}},$$
$$f = -T \ln u(0), \tag{3.3}$$

where the contour C is a loop which counterclockwise encircles the origin. Numerically, these three approaches provide perfectly the same data for free energy per site f.

4. Conclusion

Let us mention again that the main result of this paper is the calculation of $P(5)$ by means of the multi-integral representation. It is expressed only in terms of $\ln 2$ and the Riemann zeta function with odd arguments with rational coefficients. This should be the general property of $P(n)$. We expect that other correlation functions such as $\langle \boldsymbol{S}_j \boldsymbol{S}_k \rangle$ will share this property.

We have calculated numerically the value of $P(n)$ and considered the asymptotic behaviour. As was discussed in [9], the EFP $P(n)$ shows a Gaussian decay at zero temperature, while it decays exponentially at finite temperature.

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Appendix

Here we show the polynomials $P_0^{(5)}$, $P_1^{(5)}$ and $P_2^{(5)}$ which participate in the 'canonical form' T_5^c given by formula (2.5)

$$P_0^{(5)} = \tfrac{689}{18}\lambda_2\lambda_3^2\lambda_4^3\lambda_5^4 \tag{A.1}$$

$$\begin{aligned}
P_1^{(5)} = {} & \frac{796\,333}{15120}\lambda_4\lambda_5^2 - \frac{3299\mathrm{i}}{63}\lambda_1\lambda_4\lambda_5^2 + \frac{13\,844}{315}\lambda_1^2\lambda_4\lambda_5^2 - \frac{16\,517\mathrm{i}}{105}\lambda_1^3\lambda_4\lambda_5^2 - \frac{13\,463\mathrm{i}}{1512}\lambda_4\lambda_5^3 \\
& + \frac{1217\,491}{7560}\lambda_1\lambda_4\lambda_5^3 - \frac{328\,189\mathrm{i}}{840}\lambda_1^2\lambda_4\lambda_5^3 - \frac{279\,917}{945}\lambda_1^3\lambda_4\lambda_5^3 + \frac{57\,619}{630}\lambda_4^2\lambda_5^3 \\
& - \frac{543\,079\mathrm{i}}{1260}\lambda_1\lambda_4^2\lambda_5^3 - \frac{9635}{14}\lambda_1^2\lambda_4^2\lambda_5^3 + \frac{36\,199\mathrm{i}}{126}\lambda_1^3\lambda_4^2\lambda_5^3 - \frac{811\,901\mathrm{i}}{7560}\lambda_3\lambda_4^2\lambda_5^3 \\
& - \frac{392\,107}{1260}\lambda_1\lambda_3\lambda_4^2\lambda_5^3 + \frac{12\,503\mathrm{i}}{36}\lambda_1^2\lambda_3\lambda_4^2\lambda_5^3 + \frac{8}{7}\lambda_1^3\lambda_3\lambda_4^2\lambda_5^3 - \frac{197\,459}{3780}\lambda_4\lambda_5^4 \\
& - \frac{39\,169\mathrm{i}}{1512}\lambda_1\lambda_4\lambda_5^4 - \frac{20\,873}{180}\lambda_1^2\lambda_4\lambda_5^4 + \frac{66\,119\mathrm{i}}{756}\lambda_1^3\lambda_4\lambda_5^4 + \frac{66\,721\mathrm{i}}{1512}\lambda_4^2\lambda_5^4 \\
& - \frac{6301}{90}\lambda_1\lambda_4^2\lambda_5^4 + \frac{50\,921\mathrm{i}}{504}\lambda_1^2\lambda_4^2\lambda_5^4 - \frac{2045}{84}\lambda_1^3\lambda_4^2\lambda_5^4 + \frac{1474}{21}\lambda_3\lambda_4^2\lambda_5^4 \\
& - \frac{19\,549\mathrm{i}}{72}\lambda_1\lambda_3\lambda_4^2\lambda_5^4 - \frac{75\,367}{168}\lambda_1^2\lambda_3\lambda_4^2\lambda_5^4 + \frac{21\,131\mathrm{i}}{63}\lambda_1^3\lambda_3\lambda_4^2\lambda_5^4 + \frac{10\,258}{135}\lambda_4^3\lambda_5^4 \\
& - \frac{172\,369\mathrm{i}}{1512}\lambda_1\lambda_4^3\lambda_5^4 - \frac{87\,209}{504}\lambda_1^2\lambda_4^3\lambda_5^4 + \frac{21\,131\mathrm{i}}{189}\lambda_1^3\lambda_4^3\lambda_5^4 - \frac{80\,497\mathrm{i}}{504}\lambda_3\lambda_4^3\lambda_5^4 \\
& - \frac{12\,256}{27}\lambda_1\lambda_3\lambda_4^3\lambda_5^4 + \frac{72\,071\mathrm{i}}{126}\lambda_1^2\lambda_3\lambda_4^3\lambda_5^4 + \frac{42\,262}{189}\lambda_1^3\lambda_3\lambda_4^3\lambda_5^4 - \frac{161\,701}{1512}\lambda_3^2\lambda_4^3\lambda_5^4 \\
& + \frac{29\,809\mathrm{i}}{126}\lambda_1\lambda_3^2\lambda_4^3\lambda_5^4 + \frac{29\,809}{126}\lambda_1^2\lambda_3^2\lambda_4^3\lambda_5^4.
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
P_2^{(5)} = {} & -\frac{41\,011}{15\,120} + \frac{202\,861\mathrm{i}}{7560}\lambda_3 + \frac{3523}{90}\lambda_1\lambda_3 + \frac{6121}{126}\lambda_3^2 - \frac{3433\mathrm{i}}{36}\lambda_1\lambda_3^2 - \frac{99}{4}\lambda_1^2\lambda_3^2 \\
& - \frac{86\,029\mathrm{i}}{1890}\lambda_3^3 - \frac{4813}{90}\lambda_1\lambda_3^3 + \frac{130\mathrm{i}}{9}\lambda_1^2\lambda_3^3 + \frac{40}{9}\lambda_1^3\lambda_3^3 - \frac{40\,799\mathrm{i}}{1890}\lambda_5 \\
& - \frac{596\,233}{5040}\lambda_3\lambda_5 + \frac{5315\mathrm{i}}{24}\lambda_1\lambda_3\lambda_5 + \frac{582\,137\mathrm{i}}{5040}\lambda_3^2\lambda_5 + \frac{76\,147}{120}\lambda_1\lambda_3^2\lambda_5 \\
& - \frac{1640\mathrm{i}}{3}\lambda_1^2\lambda_3^2\lambda_5 + \frac{350}{9}\lambda_3^3\lambda_5 - \frac{16\,229\mathrm{i}}{36}\lambda_1\lambda_3^3\lambda_5 - \frac{2825}{4}\lambda_1^2\lambda_3^3\lambda_5 \\
& + \frac{1660\mathrm{i}}{9}\lambda_1^3\lambda_3^3\lambda_5 - \frac{2993}{56}\lambda_5^2 + \frac{1468\,283\mathrm{i}}{5040}\lambda_3\lambda_5^2 + \frac{9213}{20}\lambda_1\lambda_3\lambda_5^2 + \frac{35\,219}{112}\lambda_3^2\lambda_5^2 \\
& - \frac{28\,679\mathrm{i}}{24}\lambda_1\lambda_3^2\lambda_5^2 - \frac{6715}{8}\lambda_1^2\lambda_3^2\lambda_5^2 - \frac{15\,121\mathrm{i}}{126}\lambda_3^3\lambda_5^2 - \frac{23\,695}{36}\lambda_1\lambda_3^3\lambda_5^2 \\
& + \frac{3115\mathrm{i}}{4}\lambda_1^2\lambda_3^3\lambda_5^2 + \frac{940}{9}\lambda_1^3\lambda_3^3\lambda_5^2 + \frac{47\,321\mathrm{i}}{1260}\lambda_5^3 + \frac{209\,881}{1080}\lambda_3\lambda_5^3 - \frac{9325\mathrm{i}}{36}\lambda_1\lambda_3\lambda_5^3 \\
& - \frac{81\,509\mathrm{i}}{504}\lambda_3^2\lambda_5^3 - \frac{15\,457}{36}\lambda_1\lambda_3^2\lambda_5^3 + \frac{340\mathrm{i}}{3}\lambda_1^2\lambda_3^2\lambda_5^3 - \frac{4810}{189}\lambda_3^3\lambda_5^3 \\
& + \frac{1801\mathrm{i}}{18}\lambda_1\lambda_3^3\lambda_5^3 - \frac{3055}{18}\lambda_1^2\lambda_3^3\lambda_5^3 + 160\mathrm{i}\lambda_1^3\lambda_3^3\lambda_5^3 + \frac{6599}{378}\lambda_5^4 \\
& - \frac{1151\mathrm{i}}{21}\lambda_3\lambda_5^4 + \frac{53}{24}\lambda_1\lambda_3\lambda_5^4 - \frac{1999}{126}\lambda_3^2\lambda_5^4 - \frac{599\mathrm{i}}{4}\lambda_1\lambda_3^2\lambda_5^4 - \frac{4225}{24}\lambda_1^2\lambda_3^2\lambda_5^4 \\
& - \frac{113\mathrm{i}}{9}\lambda_3^3\lambda_5^4 - \frac{946}{9}\lambda_1\lambda_3^3\lambda_5^4 + 240\mathrm{i}\lambda_1^2\lambda_3^3\lambda_5^4 + 80\lambda_1^3\lambda_3^3\lambda_5^4.
\end{aligned} \tag{A.3}$$

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